Formulation of HMS

"symplectic Calabi-Yau"

(X,ω) compact symplectic mfd $+$ $2c_1 \simeq 0 \in H^2(X;\mathbb{Z})$

$\dim_{\mathbb{R}} X = 2n$

Contractible space $\to$ $\Gamma(X,$ topological bundle fibers homotopy $\sim S^1)$

$Sp(2n;\mathbb{R}) / \underset{\text{subgroup}}{\{g \in U(n) < Sp(2n;\mathbb{R}) \mid \det_{\mathbb{C}}(g) = \pm 1 \in \mathbb{C}^\times\}}$

$\rightsquigarrow$ Fukaya category

A_∞-Calabi-Yau of $\dim_{CY} = n$ over Novikov field

$$\mathbb{C}((q^{\mathbb{R}})) = \left\{ \sum_{i=\cdot}^{\infty} a_{E_i} q^{E_i} \;\middle|\; \begin{array}{l} E_1 < E_2 < \dots \in \mathbb{R} \\ E_i \to +\infty \\ a_{E_i} \in \mathbb{C} \end{array} \right\}$$

Version: twist by B-field $\in H^2(X, \mathbb{C}[[q^{\mathbb{R}_{\geq 0}}]]^\times)$

Contractible space of cochain representatives

Traditional definition: choice of an almost-$\mathbb{C}$ structure J ⇝ J-pseudoholomorphic discs

Objects of $\mathcal{F}(X,\omega)$: smooth (or immersed) Lagrangian $L \subset X$ (or + local system)

+ orientation issues, spin-structures ...

+ bounding cochains

compare with étale cohomology even more impossible to work with directly, ok thanks to a convenient formalism

Entrails should be hidden!

Drawbacks

- HEAVY + NOT USER-FRIENDLY
- Choice of J should be irrelevant (contractible space)
- $\mathcal{F}(X,\omega)$ is not triangulated on the nose
- $\mathcal{F}(X,\omega)$ is sometimes "too small"
  (if by Mirror Symmetry $\sim \mathrm{Perf}(\underbrace{X^\vee}_{\text{mirror}})$)

non-algebraic

Generalization to non-compact (X, ω):

choice is irrelevant

- Tame at ∞: $\exists$ complete Riemannian metric g on X

(Y. Groman 1510.04265)

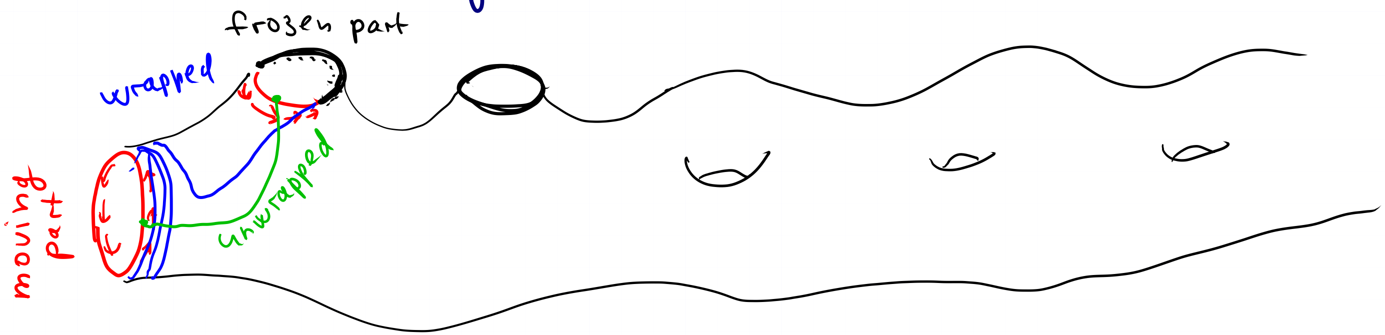
① $|\text{Curvature}(g)| \leq \text{const}$

② injectivity radius $(g) \geq \text{const}$

③ $|\text{Trace}\,(\omega^{-1}g)^2, (\omega g^{-1})^2| \leq \text{const}$

$\rightsquigarrow$ guarantees compactness of $\{$pseudo-holom. discs ...$\}$

- Partial wrapping at ∞



Goal of my talk :

Sketch of a better approach to Fukaya categories tailored to make HMS : $\mathcal{F}(X,\omega) \sim \mathrm{Perf}(X^\vee)$

PROVABLE in SYZ picture

mirror analytic variety / $\mathbb{C}((q^{\mathbb{R}}))$

+ Algebraic formalism for information-loss description of examples when $\mathcal{F}(X,\omega)$ is too small.

We are almost there!

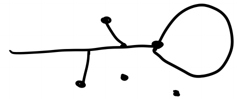
Main ingredient : Singular isotropic subsets

$\mathcal{L} \subset X$ closed

"tame" singularities, e.g. semi-$\mathbb{R}$-analytic

$\omega|_{\mathcal{L}^{smooth} \subset \mathcal{L}} = 0 \quad \Rightarrow \dim \mathcal{L} \leq n = \frac{1}{2}\dim_{\mathbb{R}} X$

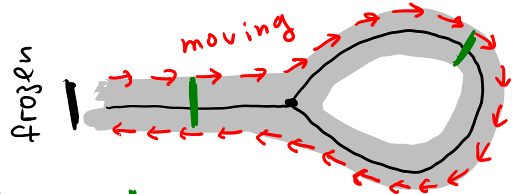
$\rightsquigarrow$ local wrapped Fukaya category $\mathcal{F}_{wr}(\mathcal{L})$

 $\subset \mathbb{R}^2$

<u>Example</u>: $\mathcal{L}$ is smooth + spin structure + $\mathcal{L} \simeq K(\Gamma, 1)$ + Maslov class $= 0 \in H^1(\mathcal{L}, \mathbb{Z})$

$\Longrightarrow \mathcal{F}_{wr}(\mathcal{L}) = \mathrm{Perf}(\mathbb{Z}\Gamma\text{-mod})$

An idea of the definition : on tubular neighborhood partial wrapping



$\mathcal{F}_{wr}(\mathcal{L}) :=$ Category generated by transversal discs at smooth pts of $\mathcal{L}$

<u>Expected</u>: $\mathcal{F}_{w.r.}(\mathcal{L})$ = global sections of a canonically defined <u>cosheaf</u> of categories

(Seifert - van Kampen - like theorem)

$n=1$: stalk at 

$= \mathrm{Perf}\,(\mathbb{Z}A_{d-1}\text{-mod})$

$\bullet \to \bullet \to \bullet \to \bullet$

$d :=$ degree of vertex

if $\mathcal{L}$ is compact $\Rightarrow$ $\mathcal{F}_{w.r.}(\mathcal{L})$ is of finite type (B.Toën-M.Vaquié)
and CY (V.Ginzburg, MK - Y.Vlassopoulos)

like $\mathrm{Perf}\,(Y^\vee)$ where $Y^\vee/\mathbb{Z}$ smooth affine
with $K_Y \simeq \mathcal{O}_Y$

D.Nadler
1507.01513 : $\mathcal{L}$ can be "deformed" to one with
a nice controllable class of "arboreal singularities"
(finite list in $\forall$ given dimension)

$\Rightarrow$ Corollary: $\mathcal{F}_{w.r.}(\mathcal{L})$ is defined $/\mathbb{Z}$ (a priori only $/\mathbb{Q}$)

Can think of as "noncommutative $CY / \mathbb{F}_1$".

Main missing ingredient: (need new techiques/ ideas)

Ambient symplectic mfd $(X, \omega) \supset \mathcal{L}$

should give canonical deformation of $\mathcal{F}_{wr}(\mathcal{L})$ $/\mathbb{C}[[t^{\mathbb{R}_{>0}}]]$

(contractible space of solutions of Maurer-Cartan eqn in cohomological Hochschild complex)

If for some almost-$\mathbb{C}$ structure J we know (by some geometric reasons) that $\nexists$ non-trivial J-holom. discs $D \subset X$ with $\partial D \subset \mathcal{L}$

$\Rightarrow$ Deformation is trivial(ized)

Example X holomorphic symplectic $\omega^{2,0}$

$\omega = \mathrm{Re}\,(e^{-i\theta}\omega^{2,0})$

$\theta \neq \mathrm{Arg}\left(\text{all} \int_\gamma \omega^{2,0} \neq 0\right)$, $\gamma \in H_2(X, \mathcal{L}; \mathbb{Z})$

cf. Y. Soibelman's talk later today

Conjecture: defomation is $/\mathbb{Z}[[t^{\mathbb{R}_{>0}}]]$

Real integrable system

$$(X,\omega) \xrightarrow{\pi} B \quad \text{base}$$

$\dim B = n$
could be singular

π is a <u>proper</u> map (fibers are compact)
all fibers are (singular) Lagrangians.

Generic fiber is $\sqcup$ of tori

<u>Assume</u> : Maslov class $= 0$

$\rightsquigarrow$ each point $b \in B$ gives deformed $\mathcal{F}_{ver}(\mathcal{L}_b)$ $\quad \mathcal{L}_b := \pi^{-1}(b)$

if $\mathcal{L}_b \simeq$ torus $(S^1)^n$: defomed algebra $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$
$= K(\mathbb{Z}^n, 1)$ $\quad$ in $\deg = 0$ (observation by M. Abouzaid)

multiplication can become noncommutative

<u>Adically convergent</u> $\quad -\varepsilon < \log|t_i| < \varepsilon$ for some $\varepsilon > 0$.

<u>Rigid-analytic tube domain</u> $\quad (\mathbb{A}^1 - 0)^n_{an} \downarrow \mathbb{R}^n \supset (-\varepsilon,\varepsilon)^n$

Anti-isoperimetric inequality (M. Gromov)

$$\text{Area}(D) \geq \text{const} \cdot \text{Length}(\partial D)$$

$D \subset X \quad \partial D \subset \mathcal{L}.$

Expect : obtain a sheaf of triangulated A_∞-categories

linear / Novikov field

or even $\mathbb{Z}((q^{\mathbb{R}}))$

on B.

At smooth point $b \in B$, $\pi^{-1}(b) =$ torus

or a noncomm. deformation.

$\text{Stalk}|_b =$ Perf (overconvergent Laurent series)

making sense as $-\varepsilon < \log|t_i| < \varepsilon$ for some $0 < \varepsilon \ll 1$.

Now it is time to speak about coverings

in (non-commutative) algebraic / analytic geometry ...

$S/_k$ separated scheme $(\mathcal{U}_i)_{i\in I}$ covering by open affine subschemes
field

($\Rightarrow$ all finite $\mathcal{U}_{i_1}\cap\ldots\cap\mathcal{U}_{i_k}$) are also affine

May assume that $(\mathcal{U}_i)_{i\in I}$ is <u>closed</u> under intersections

$\Rightarrow$ I is partially ordered

$i \leq j \Leftrightarrow \mathcal{U}_i \subseteq \mathcal{U}_j$

Quasi-coherent sheaf $\mathcal{E}$ on S

$\Leftrightarrow$ Collection of modules $\mathcal{E}_i := \Gamma(\mathcal{U}_i, \mathcal{E})$ over rings $\mathcal{O}_i := \Gamma(\mathcal{U}_i, \mathcal{O}_S)$

+ restriction maps

$\mathcal{E}_j \to \mathcal{E}_i$ for $i \leq j$ compatible with $\mathcal{O}_j$-action

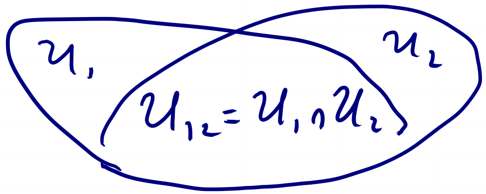
associative for $i \leq j \leq k$

+ "sheaf property": $\mathcal{O}_i \underset{\mathcal{O}_j}{\otimes} \mathcal{E}_j \to \mathcal{E}_i$ is an <u>isomorphism</u>.

Same works for unbounded derived category $D(QCoh(S))$
and for perfect complexes $Perf(S)$

Omitting "sheaf property" we obtain category of
"Čech-QCoh-presheaves"

If Covering is finite $\leftrightarrow$ A-mod for some algebra



$$A = \left\{ \begin{pmatrix} \mathcal{O}_1 & 0 & 0 \\ 0 & \mathcal{O}_2 & 0 \\ \mathcal{O}_{12} & \mathcal{O}_{12} & \mathcal{O}_{12} \end{pmatrix} \right\}$$

$$\mathcal{C} = D(QCoh(S)) \underset{\text{full}}{\subset} D(A\text{-mod})$$

Claim: $\exists$ dg algebra B + homo $A \to B$ such that $B \overset{L}{\underset{A}{\otimes}} B \to B$ is qis
(non comm. localization
e.g. $A = \mathbb{C}[t]$ $B = \mathbb{C}[t, t^{-1}]$)

$$\mathcal{C} = \{ M \in D(A\text{-mod}) \mid B \underset{A}{\otimes} M \underset{qis}{\sim} 0 \}$$

Language of pairs $(A \to B)$ of A_∞ algebras s.t.
$\mathrm{Cone}(B \otimes_A B \to B) = 0$

is very convenient

Gives <u>large</u> categories $\mathrm{Ker}(B \otimes_A \cdot)$ which are often not compactly generated
(hence "Perf" is too small)

Still, one can speak about
Hochschild (co)homology, bimodules,
derived Morita equivalence

Formalism works also for $\forall$ exact $\otimes$-category with countable $\oplus$s.
(like nuclear spaces / $\mathbb{C}$ or non-archimedean fields)

$\rightsquigarrow$ Notion of (rigid) analytic derived noncommutative space (+ quasi-coherent sheaves)

Expect (using "family Floer homology"): any <u>two</u> structures of an integrable system

$$B \xleftarrow{\pi} (X, \omega) \xrightarrow{\pi'} B'$$

give derived Morita equivalent noncomm. spaces / $\mathbb{Z}((q^{\mathbb{R}}))$